Government of **Western Australia**
Curriculum Council

WACE
Western Australian Certificate of Education

MATHEMATICS

Stage 3C/3D

WACE Examination 2011

Marking Key

Calculator-free
and
Calculator-assumed

Marking keys are an explicit statement about what the examiner expects of candidates when they respond to a question. They are essential to fair assessment because their proper construction underpins reliability and validity.

When examiners design an examination, they develop provisional marking keys that can be reviewed at a marking key ratification meeting and modified as necessary in the light of candidate responses.

Section One: Calculator Free **(40 marks)**

Question 1 **(6 marks)**

Differentiate the following with respect to x, without simplifying.

(a) $f\left(x\right)=\dfrac{4x+1}{\sqrt{x^2+1}}$ (2 marks)

Solution
$f'(x)=\dfrac{\sqrt{x^2+1}\times 4-\left(4x+1\right)\frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}}\left(2x\right)}{x^2+1}$ or $f'(x)=\dfrac{4}{\sqrt{x^2+1}}+(4x+1)(-\frac{1}{2})(x^2+1)^{-\frac{3}{2}}(2x)$
Specific Behaviours
✓ differentiates denominator of $f(x)$ correctly   ✓ combines component expressions using the quotient or product rule correctly

(b) $g\left(x\right)=xe^{x^2+1}$ (2 marks)

Solution
$g'\left(x\right)=e^{x^2+1}+xe^{x^2+1}\left(2x\right)$
Specific Behaviours
✓ differentiates correctly e^{x^2+1} ✓ combines component expressions using the product rule correctly

(c) $h\left(x\right)=\int_x^1\left(1+2t\right)^2\,dt$ (2 marks)

Solution
$h'\left(x\right)=-\left(1+2x\right)^2$ or $h(x)=\int_x^1\left(1+2t\right)^2\,dt=\left[\frac{1}{6}(1+2t)^3\right]_x^1=\frac{1}{6}\left(3\right)^3-\frac{1}{6}(1+2x)^3$ so $h'(x)=-\frac{1}{6}(3)(2)(1+2x)^2=-(1+2x)^2$
Specific Behaviours
✓ applies Fundamental Theorem of Calculus to obtain $(1+2x)^2$ ✓ recognises that the limits of integration imply that $-(1+2x)^2$ is the solution   or   ✓ integrates correctly   ✓ differentiates correctly

Question 2 **(4 marks)**

Calculate the maximum and minimum values of $x^2(6-x)$ in the interval $1 \leq x \leq 5$.

Solution
Let $f(x)=x^2(6-x)$. Since $f(x)$ is continuous, the maximum and minimum values occur at the end points of the interval or at a critical point. Now $f(x)=6x^2-x^3$, and so $f'(x)=12x-3x^2=3x(4-x)=0$ when $x=0$ or 4. So the only critical point inside the interval $1 \leq x \leq 5$ is $x=4$. Now $f(1)=1^2(6-1)=5$, $f(4)=4^2(6-4)=32$ and $f(5)=5^2(6-5)=25$. So the maximum value is 32, and the minimum value is 5.
Specific behaviours
✓ differentiates $f(x)$ correctly ✓ determines that there is a stationary point at $x=4$ ✓ evaluates $f(1)$, $f(4)$ and $f(5)$ ✓ states maximum and minimum values of $f(x)$

Question 3 **(3 marks)**

Solve the inequality

$$\frac{3x+2}{x-6}<1$$

Solution
First assume that $x>6$ Multiplying by $x-6$ gives $3x+2<x-6$, i.e. $2x<-8$, i.e. $x<-4$. This is incompatible with the assumption $x>6$, so there is no solution in the interval $x>6$. Now assume that $x<6$. Multiplying by $x-6$ gives $3x+2>x-6$, i.e. $2x>-8$, i.e. $x>-4$. Combining this with the assumption $x<6$, gives the solution $-4<x<6$. Alternatively: $\dfrac{3x+2}{x-6}-\dfrac{x-6}{x-6}<0$ $\dfrac{2x+8}{x-6}<0$ The expression on the left-hand side of this inequality can change sign only at $x=-4$ and $x=6$. Testing around these values shows that the expression is negative when $-4<x<6$.
Specific behaviours
✓ solves for the case $x>6$. ✓ solves for the case $x<6$. ✓ determines the correct interval Alternatively: ✓ rearranges inequality ✓ obtains both critical values –4 and 6 ✓ determines the correct interval

Question 4 **(4 marks)**

Let $f(x) = e^x$ and $g(x) = \sqrt{1-x}$.

(a) Determine expressions for $f(g(x))$ and $g(f(x))$. (2 marks)

Solution
$f(g(x)) = e^{\sqrt{1-x}}$ $g(f(x)) = \sqrt{1-e^x}$
Specific behaviours
✓ determines correct expression for $f(g(x))$ ✓ determines correct expression for $g(f(x))$

(b) Determine the range of $f(g(x))$. (1 mark)

Solution
Since $\sqrt{1-x}$ can be any positive number the range of $f(g(x))$ is the set of all numbers e^y where $y \geq 0.$ Since $e^0 = 1,$ the range is the interval $[1, \infty)$, or $1 \leq y < \infty$.
Specific behaviours
✓ determines the appropriate interval

(c) Determine the domain of $g(f(x))$. (1 mark)

Solution
$\sqrt{1-e^x}$ is defined provided that $1-e^x \geq 0,$ i.e. $1 \geq e^x$, i.e. $x \leq 0$ So the domain of $g(f(x))$ is the interval $x \leq 0$, or $(-\infty, 0]$.
Specific behaviours
✓ correctly determines the appropriate interval

Question 5 **(4 marks)**

(a) Evaluate $\int_{-0.5}^{0} 3(1-x)^2\,dx$ (2 marks)

Solution
$\int_{-0.5}^{0} 3(1-x)^2\,dx = \int_{-0.5}^{0}(3-6x+3x^2)dx$ $=\left[3x-3x^2+x^3\right]_{-0.5}^{0}$ or $\left[-(1-x)^3\right]_{-0.5}^{0}$ $=0-\left(3(-0.5)-3(-0.5)^2+(-0.5)^3\right)$ $=1.5+0.75+0.125=2.375$ (or $2\frac{3}{8}$)
Specific behaviours
✓ determines the antiderivative correctly ✓ substitutes correct limits of integration and evaluates integral correctly

(b) Determine $\int x^2\left(x^3+4\right)^9 dx$ (2 marks)

Solution
$\int x^2\left(x^3+4\right)^9 dx = \frac{1}{3}\int 3x^2\left(x^3+4\right)^9 dx = \frac{1}{30}\left(x^3+4\right)^{10}+c$
Specific behaviours
✓ integrates to obtain $k\left(x^3+4\right)^{10}$ ✓ determines that $k=\frac{1}{30}$

Question 6 **(6 marks)**

The cubic polynomial $p(x) = ax + bx^2 + cx^3$ has the following properties:

p(3) $= 135$

- *p(x)* has a turning point at $x = 6$
- *p(x)* has a point of inflection at $x = 2$

(a) Explain why the constants *a, b* and *c* satisfy the simultaneous equations: (3 marks)

$a + 12b + 108c = 0,\quad b + 6c = 0$ and $a + 3b + 9c = 45.$

Solution
$p(3) = a\times 3 + b\times 3^2 + c\times 3^3 = 3a + 9b + 27c = 135,$ and dividing by 3 gives $a + 3b + 9c = 45.$ $a + 3b + 9c = 45.$ $p'(x) = a + 2bx + 3cx^2$ and since *p(x)* has a turning point at $x = 6,$ $p'(6) = a + 2b\times 6 + 3c\times 6^2 = a + 12b + 108c = 0$ $p''(x) = 2b + 6cx,$ and since *p(x)* has a point of inflection at $x = 2,$ $p''(2) = 2b + 12c = 0.$ Dividing by 2 gives $b + 6c = 0.$
Specific behaviours
✓ substitutes x=3 into *p(x)* and simplifies expression ✓ differentiates *p(x)* and substitutes 6 into $p'(x)$. ✓ determines $p''(x)$, substitutes $x=2$ and simplifies.

(b) Evaluate the constants *a, b* and *c* by solving the equations in part (a). (3 marks)

Solution
Substituting $b = -6c$ in the first equation gives $a - 72c + 108c = a + 36c = 0.$ Substituting $b = -6c$ in the third equation gives $a - 18c + 9c = a - 9c = 45.$ Subtracting the last equation from the second last gives $45c = -45,$ and so $c = -1.$ Back substituting gives $a + 9 = 45$, i.e. $a = 36,$ and $b = -6\times -1 = 6.$
Specific behaviours
✓ substitutes $b = -6c$ into the other two equations ✓ solves for a ✓ back-substitutes to find b and c

Question 7 **(8 marks)**

Let S denote {1000, 1001, 1002, ..., 9998, 9999}, the set of four-digit whole numbers.

(a) How many numbers in S are palindromes (that is, read the same forward as backward) like 2002 and 7777? (2 marks)

Solution
There are 9 ways of choosing the first digit and 10 ways of choosing the second. Once these are chosen the rest are determined. So there are $9 \times 10 \times 1 \times 1 = 90$ palindromes.
Specific behaviours
✓ correctly calculates answer ✓ identifies the multiplication principle (such as $9 \times 9 \times 1 \times 1$ or $10 \times 10 \times 1 \times 1$)

(b) How many numbers in S are multiples of either 4 or 5, but not both, like 3404 and 4025? For example, 3404 is a multiple of 4 but not 5 and 4025 is a multiple of 5 but not 4. (3 marks)

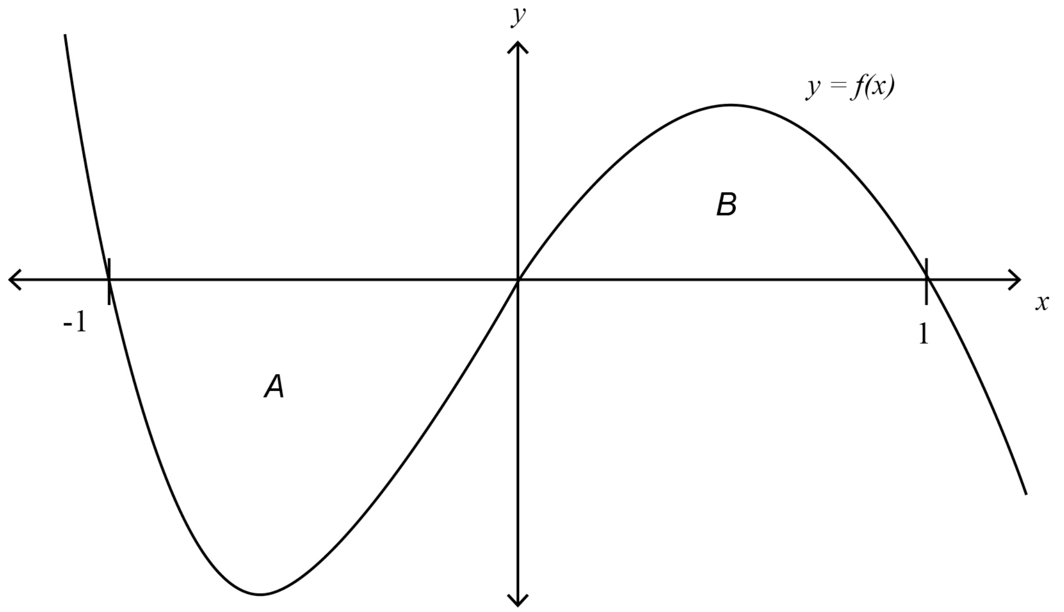
Solution
There are $9000 \div 4 = 2250$ numbers in S that are multiples of 4. There are $9000 \div 5 = 1800$ numbers in S that are multiples of 5. The total $2250 + 1800 = 4050$ counts twice the numbers that are multiples of both 4 and 5. There are $9000 \div 20 = 450$ numbers in S that are multiples of both 4 and 5, i.e. multiples of 20. So there are $4050 - 2 \times 450 = 3150$ numbers that are multiples of either 4 or 5, but not both.
Specific behaviours
✓ determines the correct number of multiples of 4 and 5 ✓ determines the correct number of multiples of 20 ✓ combines these to determine solution

(c) How many numbers in S contain at least **two (2)** consecutive 5s, like 5529, 1555 and 5255? (3 marks)

Solution
The two or more consecutive 5's must start at either the first, second or third place. There are $1 \times 1 \times 10 \times 10 = 100$ numbers in S in which the two, three or four consecutive 5's start at the first place. There are $8 \times 1 \times 1 \times 10 = 80$ numbers in S in which two or three consecutive 5's start at the second place. The first digit cannot be 0 or 5. There are $9 \times 9 \times 1 \times 1 = 81$ numbers in S in which two consecutive 5's start at the third place. The first digit cannot be 0 and the second cannot be 5. So there are $100 + 80 + 81 = 261$ such numbers in S.
Specific behaviours
✓ shows that the first digit must not be zero ✓ splits the set of numbers satisfying the condition into manageable subsets ✓ obtains correct answer by combining these subsets appropriately

Question 8 **(5 marks)**

Part of the graph of $y = f(x)$ is shown below. The areas of the bounded regions *A* and *B* are 7 and 4 square units respectively.



(a) Evaluate $\int_0^1 f(-x)dx$ (2 marks)

Solution
The graph of $y = f(-x)$ is the reflection of the above graph in the y-axis. The required integral is the signed area $-A$, i.e. -7 .
Specific behaviours
✓ determines correctly the magnitude of the integral ✓ determines correctly the sign of the integral

(b) Evaluate $\int_{-1}^1 (2 - f(x))dx$ (3 marks)

Solution
$\int_{-1}^{1}\left(2-f\left(x\right)\right)dx=\int_{-1}^{1}2\,dx-\int_{-1}^{1}f\left(x\right)dx=4-\left(-7+4\right)=7$
Specific behaviours
✓ identifies correctly the two components of the integral ✓✓ determines the two integrals

Section Two: Calculator assumed **(80 marks)**

Question 9 **(5 marks)**

For events A and B;

$$P(A) = 0.5, \quad P(B \mid A) = 0.3 \quad \text{and} \quad P(A \cup B) = 0.8.$$

(a) Calculate $P(A \cap B)$ (1 mark)

Solution
$P(A \cap B) = P(B\|A)P(A) = 0.3 \times 0.5 = 0.15$
Specific behaviours
✓ calculates answer correctly

(b) Calculate $P(B)$ (1 mark)

Solution
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$, That is, $0.8 = 0.5 + P(B) - 0.15$, and so $P(B) = 0.8 - 0.5 + 0.15 = 0.45$
Specific behaviours
✓ calculates answer, consistent with answer from (a)

(c) Calculate $P(\bar{A} \cap B)$ (1 mark)

Solution
$P(\bar{A} \cap B) + P(A \cap B) = P(B)$ That is, $P(\bar{A} \cap B) + 0.15 = 0.45$ and so $P(\bar{A} \cap B) = 0.3$
Specific behaviours
✓ calculates answer, consistent with answers from (a) and (b)

(d) Are events A and B independent? Justify your answer. (2 marks)

Solution
They are not independent, since $P(B) = 0.45$, while $P(B\|A) = 0.3 \neq P(B)$
Specific behaviours
✓ states a condition for independence, showing correct numerical values ✓ identifies that A and B are not independent

Question 10 **(8 marks)**

Suppose that 9% of people in a certain community live alone.

Sixty percent of people who live alone own pets, whereas only 3% of people who do not live alone own pets.

(a) What is the probability that a person chosen at random from this community owns a pet? (2 marks)

Solution
Let C denote the event that a randomly chosen person lives alone, and D denote the event that a randomly chosen person owns a pet. Then $P(C) = 0.09,\ P(D\vert C) = 0.6,$ and $P(D\vert\bar{C}) = 0.03$ So $P(D) = P(D \cap C) + P(D \cap \bar{C})$ $= P(D\vert C)P(C) + P(D\vert\bar{C})P(\bar{C})$ $= 0.6 \times 0.09 + 0.03 \times 0.91 = 0.054 + 0.0273 = 0.0813$
Specific behaviours
✓ correctly determines at least one of $P(D\vert C)$or $P(D\vert\bar{C})$ ✓ determines probability of owning a pet

(b) What is the probability that a randomly-selected person from this community neither lives alone nor owns a pet? (1 mark)

Solution
$P(\bar{C} \cap \bar{D}) = P(\bar{C}) \times P(\bar{D} \vert \bar{C}) = 0.91 \times 0.97 = 0.8827$
Specific behaviours
✓ calculates correct answer

(c) What percentage of people from this community who own a pet live alone? (2 marks)

Solution
$P(C\vert D) = \dfrac{P(C \cap D)}{P(D)} = \dfrac{0.054}{0.0813} \approx 0.664 = 66.4\%$
Specific behaviours
✓ identifies conditional probability with answer to (a) as denominator ✓ determines correct percentage

(d) In a group of 30 people in this community, four live alone. If six people are selected from this group, what is the probability that no more than two of them live alone? (3 marks)

Solution
$$P = \frac{\binom{4}{0}\binom{26}{6} + \binom{4}{1}\binom{26}{5} + \binom{4}{2}\binom{26}{4}}{\binom{30}{6}} = 0.982$$
Specific behaviours
✓ determines the expression for the denominator ✓ determines a numerator which is at least partly correct ✓ determines the probability

Question 11 **(5 marks)**

When an amount $\$A$ is invested at an interest rate of $r\%$ per annum, compounded n times per year, the value $\$V$ of the investment after one year is given by $V = A\left(1+\frac{r}{100n}\right)^n$.

Kelvin invests $6000 at 8% per annum interest for one year.

(a) What is the value of the investment at the end of the year if interest is compounded twice per year? (1 mark)

Solution
$V = 6000\left(1+\frac{8}{2\times 100}\right)^2 = 6489.60$ So the value at the end of the year is $6489.60
Specific behaviours
✓ evaluates correct value of investment

(b) If interest is compounded monthly, by what percentage does the investment increase over the course of the year? (2 marks)

Solution
$V = 6000\left(1+\frac{8}{12\times 100}\right)^{12} = 6498.00$ So the percentage growth is $\frac{6498-6000}{6000}\times 100 = 8.3\%$
Specific behaviours
✓ calculates correct value of investment after one year ✓ calculates correct percentage increase

(c) If the investment were to be compounded more frequently, could the value of Kelvin's investment rise above $6500 at the end of the year? Justify your answer. (2 marks)

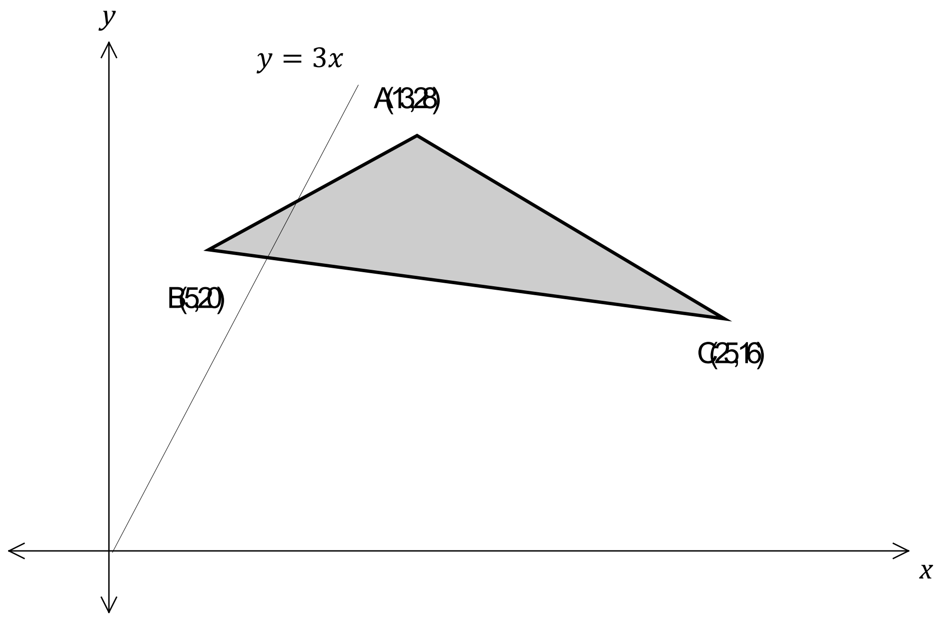
Solution
The maximum occurs if interest is compounded 'continuously'. In which case the value after one year is given by $V = 6000e^{0.08} = 6499.72$ Therefore the value can't rise above $6500 by more frequent compounding.
Specific behaviours
✓ calculates the correct limiting value ✓ interprets the limiting value

Question 12 **(9 marks)**

Each day, a company produces x thousand units of commodity X and y thousand units of commodity Y.

Each unit of commodity X earns a profit of \$21, and each unit of commodity Y earns a profit of \$15.

The feasible region for the company's daily production schedule is the triangle shown below.



(a) Determine the inequality satisfied by x and y that corresponds to the edge AB of the feasible region. (2 marks)

Solution
The equation of line AB is $y = x + 15$ The inequality is $y \le x + 15$
Specific behaviours
✓ determines correct equation ✓ states correct inequality

(b) Determine the maximum possible daily profit. (2 marks)

Solution
Let P denote the daily profit, in thousands of dollars. Then $P = 15x + 21y$. The maximum value occurs at a corner point: At A, $P = 21 \times 13 + 15 \times 28 = 693$ At B, $P = 21 \times 5 + 15 \times 20 = 405$ At C, $P = 21 \times 25 + 15 \times 16 = 765$ So the maximum daily profit is \$765 000.
Specific behaviours
✓ evaluates profit at vertices ✓ draws correct conclusion, in thousands of dollars

(c) The company decides that the amount of commodity Y produced cannot be more than three times the amount of commodity X.

How does this additional constraint affect the maximum possible profit? Justify your answer. (2 marks)

Solution
The new constraint is $y \leq 3x$ (Indicated on graph) Since the corner point C satisfies this constraint, the maximum possible profit is not affected.
Specific behaviours
✓ describes new constraint in written form or as a line on the diagram ✓ interprets effect of constraint, with reasoning correct interpretations with inaccurate reasons (eg stating that the new constraint does not intersect the feasible region) will be awarded one mark.

(d) Changing market conditions will cause changes to the unit profits for commodities X and Y.

On each of the next seven days, the unit profit for commodity X will fall by \$1, and the unit profit for commodity Y will rise by \$1.

Describe how these changes will affect the maximum possible daily profit over the course of the next week. (3 marks)

Solution
On day n the profit is $P = (21 - n)x + (15 + n)y$ Initially, the maximum profit is obtained at vertex C The profit at C is $25(21 - n) + 16(15 + n) = 765 - 9n$ So the profit will fall by \$9 000 per day initially The profit at A is $13(21 - n) + 28(15 + n) = 693 + 15n$ The profit at C will be equal to the profit at A when $765 - 9n = 693 + 15n$, that is, when $n = 3$. After three (3) days, the profit will be maximised at vertex A, so it will then start rising by \$15 000 per day.
Or alternative solution
After day 1, maximum profit will drop to \$756 000 After day 2, maximum profit will drop to \$747 000 After day 3, maximum profit will drop to \$738 000 Then After day 4, maximum profit will rise to \$753 000 After day 5, maximum profit will rise to \$768 000 After day 6, maximum profit will rise to \$783 000 After day 7, maximum profit will rise to \$798 000 (using CAS spreadsheet)
Specific behaviours
✓ identifies initial fall in profits ✓ identifies that profits will then rise ✓ calculates that the change from falling to rising profits occurs when $n = 3$

Question 13 **(7 marks)**

The lifetimes of Glowbrite light bulbs are normally distributed with mean 3500 hours and standard deviation 200 hours.

(a) What is the probability that the lifetime of a randomly-selected bulb is at least 3400 hours? (1 mark)

Solution
Let T hours denote the lifetime of a random Glowbrite light bulb. Then T has a normal distribution with mean $\mu = 3500$ and standard deviation $\sigma = 200$. Then $P(T > 3400) = 0.6915$ (from calculator)
Specific behaviours
✓ calculates correct answer

(b) Calculate t, given that 5% of Glowbrite bulbs last longer than t hours. (1 mark)

Solution
If $P(T > t) = 0.05$, $t = 3829$ (from calculator)
Specific behaviours
✓ calculates correct answer

(c) What is the probability that a Glowbrite bulb will last no more than 3500 hours, if it has already lasted 3200 hours? (3 marks)

Solution
$P(T \leq 3500 \mid T \geq 3200) = \dfrac{P(3200 \leq T \leq 3500)}{P(T \geq 3200)} = \dfrac{0.9332 - 0.5}{0.9332} = 0.4642$
Specific behaviours
✓ indicates correct conditional probability required (symbolically or in words) ✓ determines correct denominator ✓ calculates probability correctly

(d) The company also produces Ultrabrite light bulbs, whose lifetimes are also normally distributed, with a possibly different mean μ hours but the same standard deviation 200 hours.

A quality control expert at the company wishes to estimate μ using the mean lifetime of a random sample of Ultrabrite bulbs.

How large should the sample be in order to be 95% confident that the estimate will be no more than 10 hours in error? (2 marks)

Solution
The 95% confidence interval for μ is $\bar{x} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}$ i.e. $\lvert\bar{x} - \mu\rvert \leq z\frac{\sigma}{\sqrt{n}} = 1.96\frac{\sigma}{\sqrt{n}}$. So we want $1.96\frac{\sigma}{\sqrt{n}} = 10$, i.e. $n = \left(\frac{1.96\sigma}{10}\right)^2 = \left(\frac{1.96\times 200}{10}\right)^2 = 1536.64$ So the sample size needs to be at least 1537.
Specific behaviours
✓ states the correct equation for a 95% confidence interval ✓ calculates the correct sample size

Question 14 **(5 marks)**

During a volcanic eruption a rock is ejected from the top of the volcano. The rock rises upward and then falls onto a flat plain 1500 metres below the top of the volcano. During its flight, the vertical velocity of the rock, v m/s, is given by

$$v = 160 - 9.8t$$

where t seconds is the time after the ejection of the rock.

(a) How high does the rock rise above the top of the volcano? (3 marks)

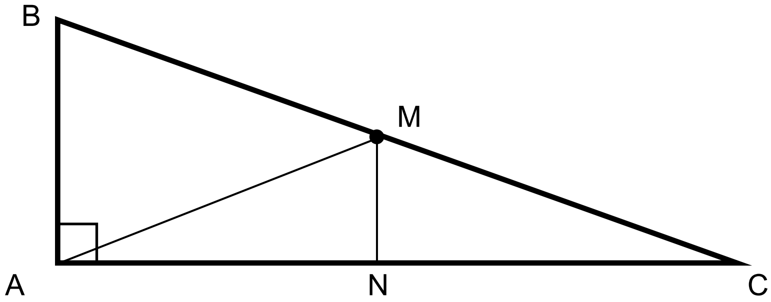
Solution
Let $y(t)$ m be the height of the rock above the top of the volcano at time t. Then $y = \int_0^t (160 - 9.8u)du = (160u - 4.9u^2)\|_{u=0}^{u=t} = 160t - 4.9t^2$. At the highest point $v = 0$, i.e. $160 - 9.8t = 0$, and $t = 160/9.8 \cong 16.327$ So $y_{max} = 160 \times 16.327 - 4.9 \times 16.327^2 = 1306.12$ So the rock rises to the impressive height of 1306.12 metres above the top of the volcano. Alternatively, using a formula for rectilinear motion: If $v^2 = u^2 + 2as$, with $v = 0$, $u = 160$ and $a = -9.8$, $s = 1306.12$
Specific behaviours
✓ integrates to find an equation for the height ✓ calculates the time at which the height is a maximum ✓ calculates the maximum height

(b) How long does it take for the rock to reach the plain below? (2 marks)

Solution
If $y = -1500$, we have $-1500 = 160t - 4.9t^2$. So $t \cong 40.26$ (by calculator) So it takes 40.26 seconds for the rock to reach the plain below.
Specific behaviours
✓ determines correct equation ✓ solves equation to find time

Question 15 **(4 marks)**

In the diagram below ABC is a right-angled triangle, and M is the mid-point of the hypotenuse BC.



Prove that M is equidistant from each of the vertices A, B and C.

Hint: Start by drawing the line through M that is parallel to the side AB

Solution
For triangles ABC and NMC, $\angle ACB$ is the common angle Since MN is parallel to AB, the size of $\angle CMN$ is equal to the size of $\angle CBA$ (and the size of $\angle CNM$ is equal to the size of $\angle CAB$) Therefore triangle ABC and triangle NMC are similar. (AA similarity) By similarity $\frac{AC}{NC} = \frac{BC}{MC} = 2$ and so $\mathrm{AC} = 2\mathrm{NC}$. Hence $AN = NC$. Therefore triangles ANM and CNM are congruent (SAS) (equal sides AN and NC, common side NM and included angles $\angle ANM$ and $\angle CNM$ are both right angles). So $MA = MC$ So $MA = MB = MC$
Specific behaviours
✓✓ proves that Δ ABC and Δ NMC are similar(AA). ✓✓ proves that Δ ANM and Δ CNM are congruent (SAS) Note: Proofs which are generally correct but that contain some inappropriate reasoning will be awarded one mark

Question 16 **(11 marks)**

The mean μ and standard deviation σ of the uniform distribution on the interval [a, b] are given by

$$\mu = \frac{a+b}{2} \text{ and } \sigma = \frac{b-a}{2\sqrt{3}}.$$

A calculator can generate random numbers that are uniformly distributed between 0 and 1.

(a) For this distribution of the random numbers generated by the calculator, calculate

(i) the mean. (1 mark)

(ii) the standard deviation (to **three (3)** decimal places). (1 mark)

Solution
Mean = 0.5 Standard deviation = 0.289
Specific behaviours
✓ calculates mean ✓ calculates standard deviation to three (3) decimal places

(b) What is the probability that a randomly-generated number lies between $\frac{1}{4}$ and $\frac{1}{3}$?
(1 mark)

Solution
$P = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} = 0.08\dot{3}$
Specific behaviours
✓ calculates probability correctly

(c) What is the probability that a randomly-generated number contains no seven in its first five (5) decimal places? (1 mark)

Solution
$P = 0.9^5 = 0.59049$
Specific behaviours
✓ calculates correct answer

(d) What is the probability that a randomly-generated number contains at most three odd digits in its first five (5) decimal places? Give your answer to **four (4)** decimal places. (2 marks)

Solution
The probability of each digit being odd is 0.5. Let X be the number of odd digits in the first 5 decimal places. Then X ~ Bin (5, 0.5) and P (X ≤ 3) = 0.8125.
Specific behaviours
✓ identifies binomial distribution with parameters ✓ calculates correct probability

(e) What is the probability that the sum of 500 randomly-generated numbers exceeds 260? Give your answer to **four (4)** decimal places. (3 marks)

Solution
P (sum > 260) = P (mean > 0.52) The sampling distribution of the mean can be expected, by the central limit theorem, to be normal. The mean = 0.5 and standard deviation is $\frac{0.289}{\sqrt{500}} = 0.0129$ So $P(\text{Mean} > 0.52) = 0.0607$
Specific behaviours
✓ identifies that the mean must be greater than 0.52 ✓ states the distribution and parameters of the sampling distribution of the mean ✓ calculates correct probability (to four decimal places)

(f) Another uniform distribution on an interval $[a,b]$ has a standard deviation of $2\sqrt{3}$. How wide is the interval? (2 marks)

Solution
$\frac{b-a}{2\sqrt{3}} = 2\sqrt{3}$ so $b-a = \left(2\sqrt{3}\right)^2 = 12$
Specific behaviours
✓ substitutes $\sigma = 2\sqrt{3}$ into formula ✓ calculates width of interval correctly

Question 17 **(7 marks)**

Let A denote the average of the squares of two numbers x and y:

$$A = \frac{1}{2}\left(x^2 + y^2\right),$$

and let B denote the square of the average of x and y:

$$B = \left(\frac{x+y}{2}\right)^2 .$$

(a) Evaluate A and B in the case $x = 5$ and $y = 7$. (1 mark)

Solution
$A = \frac{1}{2}(5^2 + 7^2) = \frac{1}{2}(25 + 49) = 37$, and $B = \left(\frac{5+7}{2}\right)^2 = 6^2 = 36$
Specific behaviours
✓ calculates both A and B

(b) What can be said about A and B if $x = y$? Justify your answer. (2 marks)

Solution
$A = \frac{1}{2}(x^2 + x^2) = x^2$, and $B = \left(\frac{x+x}{2}\right)^2 = x^2$ So $A = B$ if $x = y$
Specific behaviours
✓ states that A and B are equal ✓ justifies answer algebraically

(c) Examine the difference between A and B for various values of x and y and state a conjecture about A and B. (1 mark)

Solution
$A \geq B$ for all x and y
Specific behaviours
✓ states correct conjecture

(d) Prove the conjecture in part (c). (3 marks)

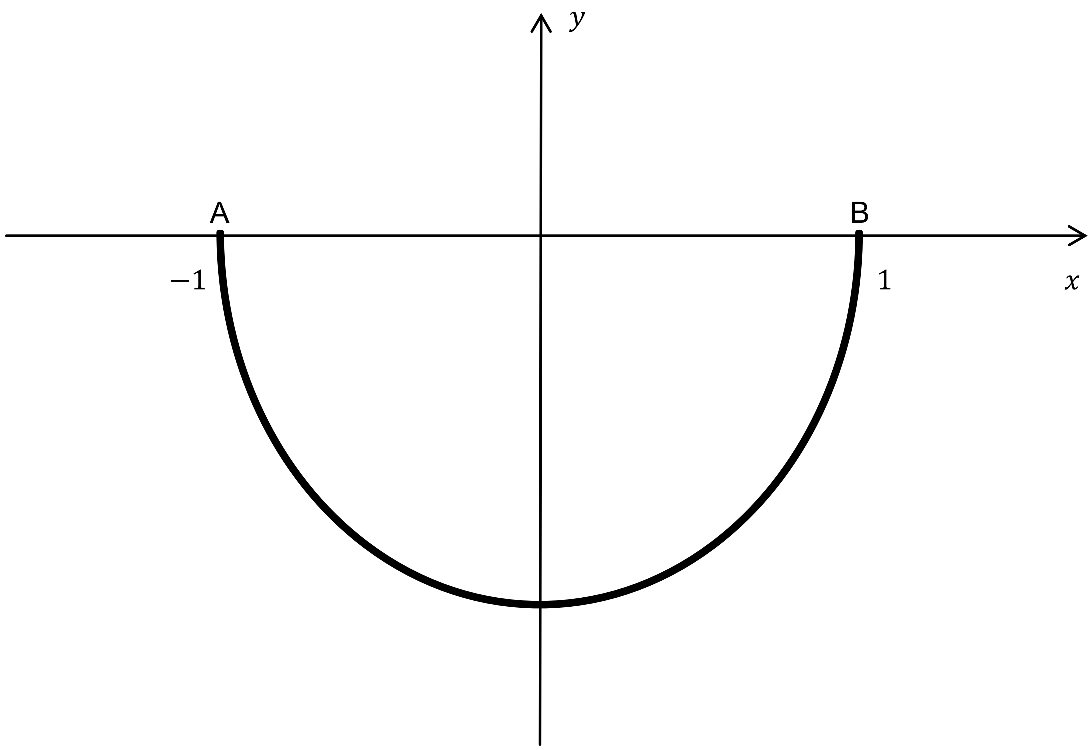
Solution
$A - B = \frac{1}{2}(x^2 + y^2) - \left(\frac{x+y}{2}\right)^2 = \frac{1}{2}(x^2 + y^2) - \frac{1}{4}(x + y)^2$ $= \frac{1}{4}(2x^2 + 2y^2 - (x + y)^2) = \frac{1}{4}(2x^2 + 2y^2 - x^2 - 2xy - y^2)$ $= \frac{1}{4}(x^2 - 2xy + y^2) = \frac{1}{4}(x - y)^2 \geq 0$
Or alternative solution
Alternative proof (which could be provided with aid of a CAS calculator) $A - B = \frac{1}{2}(x^2 + y^2) - \left(\frac{x + y}{2}\right)^2 = \frac{1}{2}(x^2 + y^2) - \frac{1}{4}(x + y)^2$ $= \left(\frac{x-y}{2}\right)^2$ ≥ 0
✓ states an expression for $A - B$ ✓ simplifies this expression   ✓ factorises to reveal that expression is non-negative

Question 18 **(7 marks)**

A cable hanging between two points $A(-1,0)$ and $B(1,0)$ lies on the curve

$$y = e^{cx} - d + e^{-cx},$$

where c and d are positive constants.



(a) Show that $d = e^{c} + e^{-c}$. (1 mark)

Solution
$y = 0$ when $x = 1$, and so $0 = e^{c} - d + e^{-c}$ Hence $d = e^{c} + e^{-c}$.
Specific behaviours
✓ substitutes (–1,0) or (1,0) into equation

(b) Use calculus to show that the lowest point on the cable occurs where it crosses the y-axis, that is, where $x = 0$. (3 marks)

Solution
The minimum will occur when $\frac{dy}{dx} = 0$ Now $\frac{dy}{dx} = ce^{cx} - ce^{-cx} = c(e^{cx} - e^{-cx}) = 0$ This occurs when $e^{cx} = e^{-cx}$. Dividing both sides by e^{-cx}, we have $e^{2cx} = 1$ So $x = 0$.
Specific behaviours
✓ calculates derivative ✓ simplifies to find that $e^{cx} = e^{-cx}$ ✓ solves for x

(c) The length s of the curve $y = f(x)$, between the limits $x = a$ and $x = b$ is given by the formula

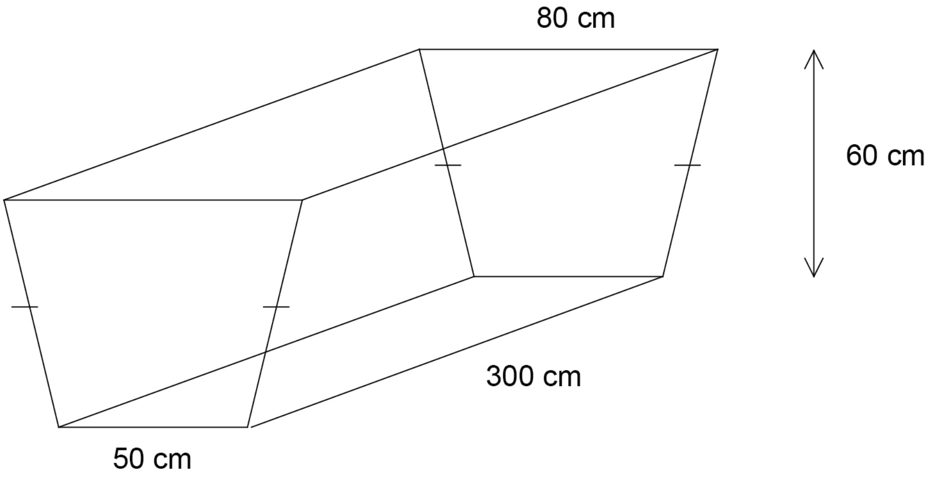
$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Use this formula to determine the length of the cable if the lowest point of the cable is 10 units below the level of the supports A and B. (3 marks)

Solution
Since the minimum occurs at $x = 0$, $-10 = e^0 - e^c - e^{-c} + e^0$, i.e. $e^c + e^{-c} = 12$. From the calculator: $c = 2.47789$ So $\frac{dy}{dx} = 2.47789(e^{2.477889x} - e^{-2.477889x})$, So $s = \int_{-1}^{1} \sqrt{1 + \left(2.47789(e^{2.47789x} - e^{-2.47789x})\right)^2}\, dx = 20.2712$
Specific behaviours
✓ states equation which can be used to find c ✓ solves for c ✓ calculates length of curve

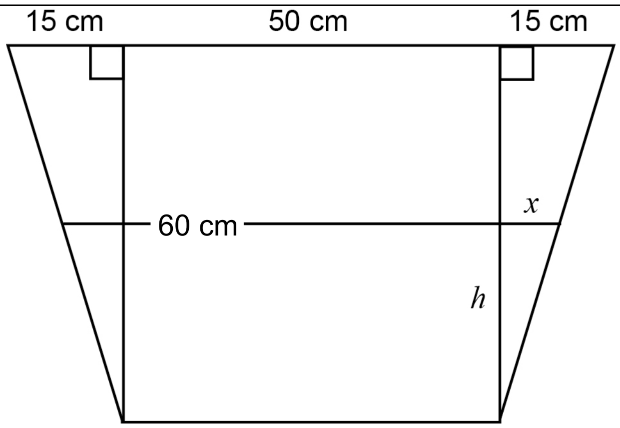
Question 19 **(8 marks)**

A water trough has the shape of a trapezoidal prism. It is 300 centimetres long, 60 centimetres high, 80 centimetres wide at the top and 50 centimetres wide at the bottom. A sketch of the trough is shown below (not to scale).



(a) The top surface of the water in the trough has the shape of a rectangle whose length is 300 cm. Show that if the water in the trough is h cm deep, then the width of this rectangle is $50+0.5h$ cm. (2 marks)

Solution



In the similar triangles

$$\frac{x}{h}=\frac{15}{60}$$

$$x=0.25h$$

$\therefore$ Width of the rectangle $=50+2x$

$$=50+2(0.25h)$$

$$=50+0.5h \text{ cm}$$

Specific behaviours

✓ determines an equation for x using similar triangles
✓ determines the width of the rectangle

Alternative solution

Solution
The width w cm of rectangular top surface of the water is a linear function of the depth of water h. Since $w = 50$ when $h = 0$ and $w = 80$ when $h = 60$, $w = 50 + \frac{80-50}{60}h$ $= 50 + 0.5h$
Specific behaviours
✓ determines rate at which width is increasing with respect to height $\left(\frac{80-50}{60}\right)h = 0.5h$ ✓ determines formula for width

(b) Show that if water in the trough is h cm deep, then the volume of water, V litres, is given by $V = h(15 + 0.075h)$ (2 marks)

If the depth is h cm the vertical cross section of the water is a trapezoid with height h, bottom width 50 cm and top width $50 + 0.5h$ cm. So the cross-sectional area is $h \times \frac{1}{2}(50 + 50 + 0.5h) = h(50 + 0.25h)\ \mathrm{cm}^2$, and the volume is $300h(50 + 0.25h)\ \mathrm{cm}^3$ Since 1 litre $= 1000\ \mathrm{cm}^3$, the volume is $\frac{300}{1000}h(50 + 0.25h) = h(15 + 0.075h)$ litres
Specific behaviours
✓ determines correct expression for cross-sectional area of water ✓ determines correct expression for volume in litres

(c) Water is being pumped into the trough at a rate of 40 litres per minute. At what rate is the depth of the water increasing at the instant when the trough is half full by volume? Give your answer correct to the nearest centimetre per minute. (4 marks)

Solution
The capacity of the tank is $60(15 + 0.075 \times 60) = 1170$ litres. Half-capacity is 585 litres. If $h(15 + 0.075h) = 585$, $h = 33.4$ Now $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} = (15 + 0.15h) \times \frac{dh}{dt}$ Since $\frac{dV}{dt} = 40$ and $h = 33.4$ $40 = (15 + 0.15 \times 33.4) \times \frac{dh}{dt}$ i.e. $\frac{dh}{dt} = 1.9999$ So the depth is increasing at the rate of 2 centimetres per minute.
Specific behaviours
✓ calculates the volume of the trough when it is half full ✓ solves to find h at this time ✓ finds $\frac{dV}{dh}$ ✓ calculates rate of change of depth to nearest centimetre per minute

Question 20 **(4 marks)**

(a) Find the volume of revolution obtained when the line $y = mx$, between the limits $y = 0$ and $y = h$, is rotated about the y–axis. (2 marks)

Solution
$V = \pi \int_0^h x^2 dy = \pi \int_0^h \left(\frac{y}{m}\right)^2 dy = \frac{\pi h^3}{3m^2}$
Specific behaviours
✓ substitutes $x = \frac{y}{m}$ into formula for volume of solid of revolution ✓ integrates to find volume

(b) Use your answer from part (a) to show that the volume V of a cone is given by

$$V = \frac{1}{3}Ah$$

where A is the area of the base and h is the height. (2 marks)

Solution
Since $y = mx$, $x = h/m$ when $y = h$. So the radius of the base is h/m. Therefore the area of the base is given by $A = \pi\left(\frac{h}{m}\right)^2$ So $V = \frac{\pi h^3}{3m^2} = \frac{1}{3}\pi\left(\frac{h}{m}\right)^2 h = \frac{1}{3}Ah$
Specific behaviours
✓ determines correct expression for the area of the base of the cone ✓ shows that volume from part (a) = $\frac{1}{3}h \times$ area of base